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On vector variational-like inequality problems ☆

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Abstract

In this paper, we establish some relationships between vector variational-like inequality and vector optimization problems under the assumptions of α-invex functions. We identify the vector critical points, the weakly efficient points and the solutions of the weak vector variational-like inequality problems, under pseudo-α-invexity assumptions. These conditions are more general than those of existing ones in the literature. In particular, this work extends the earlier work of Ruiz-Garzon et al. [G. Ruiz-Garzon, R. Osuna-Gomez, A. Rufian-Lizan, Relationships between vector variational-like inequality and optimization problems, European J. Oper. Res. 157 (2004) 113–119] to a wider class of functions, namely the pseudo-α-invex functions studied in a recent work of Noor [M.A. Noor, On generalized preinvex functions and monotonicities, J. Inequal. Pure Appl. Math. 5 (2004) 1–9].

Keywords: Vector optimization; Pseudo-univex functions; Vector variational-like inequality problems

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1. Introduction

The concept of vector variational inequality was introduced by Giannessi [1] in 1980. Since it has shown applications to a wide range of problems in various disciplines in the natural and social sciences, vector variational inequality problems have been generalized in various directions; in particular, vector variational-like inequality problems, see [2,12,14,16].

The role of generalized monotonicity of the operator in vector variational inequality problems corresponds to the role of generalized convexity of the objective function in the optimization problem. In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson [3]. For further developments in this direction see [4–6,11]. Weir and Mond [13] and Noor [7–9] have studied some basic properties of the preinvex functions and their role in optimization and variational-like inequality problems. Noor [8] has pointed out that the concept of invexity plays exactly the same role in variational-like inequality problems as the classical convexity plays in variational inequality problems, and has shown that the variational-like inequality problems are well defined in the setting of invexity.

Recently, Ruiz-Garzon et al. [12] established relationships between vector variational-like inequality and optimization problems, under the assumptions of pseudo-invexity. However, Ruiz-Garzon et al. [12] have obtained some results without invexity assumption on the underlying set while discussing variational-like inequality problems.

Recently Noor [9] has studied some properties of the *α-preinvex* functions and their differentials. Motivated by the work of Noor [9], we establish various relationships between generalized vector variational-like inequality problems and vector optimization problems under the assumption of pseudo-α-invex functions. In Section 2, we recall some definitions and preliminaries. In Section 3, we establish relationships between generalized vector variational-like inequality problems and vector optimization problems.

2. Preliminaries

The following convention for equalities and inequalities will be used throughout the paper. If $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in R^n$, we denote

$$
\begin{aligned}
&x \leqq y \quad \text{iff} \quad x_i \leqslant y_i \quad \forall i = 1, 2, \ldots, n;\\
&x \leq y \quad \text{iff} \quad x_i \leqslant y_i \quad \forall i = 1, 2, \ldots, n \text{ with } x \neq y;\\
&x < y \quad \text{iff} \quad x_i < y_i \quad \forall i = 1, 2, \ldots, n; \quad \text{and}\\
&x \not< y \text{ is the negation of } x < y.
\end{aligned}
$$

Let X be a nonempty subset of R^n, $\eta : X \times X \to R^n$ be a continuous map and $\alpha : X \times X \to R_+ \setminus \{0\}$ be a bifunction. First of all, we recall some known results and concepts, see Noor [9].

Definition 2.1. A subset X is said to be an α-invex set, if there $\eta : X \times X \to R^n$, $\alpha(x,u) : X \times X \to R_+$ such that

$$u + \lambda\alpha(x,u)\eta(x,u) \in X, \quad \forall x, u \in X,\ \lambda \in [0,1].$$

It is well known that the α-*invex* set may not be convex sets, see Noor [9].

From now onward we assume that the set X is a nonempty, closed and α-invex set with respect to $\alpha(\cdot,\cdot)$ and $\eta(\cdot,\cdot)$, unless otherwise specified.

Definition 2.2. The function f on the α-invex set is said to be α-*preinvex* function, if there exist $\eta : X \times X \to R^n$, $\alpha(x,u) : X \times X \to R_+$ such that

$$f\big(u + \lambda\alpha(x,u)\eta(x,u)\big) \leqslant (1-\lambda)f(u) + \lambda f(x), \quad \forall x, u \in X,\ \lambda \in [0,1].$$

Definition 2.3. Let $f : X \subset R^n \to R^p$ be a differentiable function with a $p \times n$ matrix as its Jacobian. The function f is said to be

(a) α-invex if and only if there exist functions $\alpha : X \times X \to R_+ \setminus \{0\}$ and $\eta : X \times X \to R^n$, such that

$$f(y) - f(x) \geqq \big\langle \alpha(y,x)\nabla f(x), \eta(y,x)\big\rangle, \quad \forall x, y \in X;$$

(b) strictly α-invex if and only if there exist functions $\alpha : X \times X \to R_+ \setminus \{0\}$ and $\eta : X \times X \to R^n$, such that

$$f(y) - f(x) > \big\langle \alpha(y,x)\nabla f(x), \eta(y,x)\big\rangle, \quad \forall x, y \in X,\ x \neq y;$$

(c) pseudo-α-invex if and only if there exist functions $\alpha : X \times X \to R_+ \setminus \{0\}$ and $\eta : X \times X \to R^n$, such that

$$f(y) < f(x) \quad \Rightarrow \quad \big\langle \alpha(y,x)\nabla f(x), \eta(y,x)\big\rangle < 0, \quad \forall x, y \in X.$$

Note that if $\alpha(y,x) = 1$, then Definitions 2.1–2.3 reduces to the one in [3,11,13].

Let $X \subseteq R^n$ be an α-invex nonempty subset of R^n and two continuous maps $F : X \to R^n$ and $\eta : X \times X \to R^n$ and $\alpha : X \times X \to R_+ \setminus \{0\}$ be a bifunction.

The variational-like inequality problem (VLIP) is to find a point $\bar{x} \in X$ such that $\eta(y,x)^{\mathrm{T}} F(\bar{x}) \geqslant 0$, $\forall y \in X$.

A vector variational-like inequality problem (VVLIP), is to find a point $\bar{x} \in X$, such that there exists no $y \in X$, such that $F(\bar{x})\eta(y,\bar{x}) \leqslant 0$.

A weak vector variational-like inequality problem (WVVLIP), is to find a point $\bar{x} \in X$, such that there exists no $y \in X$, such that $F(\bar{x})\eta(y,\bar{x}) < 0$.

We consider the following generalized forms of vector variational-like inequality problems:

(GVVLIP) A generalized vector variational-like inequality problems, is to find a point $\bar{x} \in X$, such that there exists no $y \in X$, such that $\langle \alpha(y,\bar{x})F(\bar{x}), \eta(y,\bar{x})\rangle \leqslant 0$.

(GWVVLIP) A generalized weak vector variational-like inequality problems, is to find a point $\bar{x} \in X$, such that there exists no $y \in X$, such that $\langle \alpha(y,\bar{x})F(\bar{x}), \eta(y,\bar{x})\rangle < 0$.

Remark 2.1. Notice that if $\alpha(y,\bar{x}) = 1$, then the (GVVLIP) and (GWVVLIP) reduce to the (VVLIP) and (WVVLIP) studied in Garzon et al. [12].

It is well known that in multi-objective optimization problems, the objective functions are conflicting in nature and cannot be combined into a single objective. In this sense we must understand the concept of efficient solutions.

Let $f : R^n \to R^p$, the vector optimization problem (VOP) is to find the *efficient points* for

$$\text{(VOP)} \quad \text{V-min}\, f(x) \quad \text{subject to} \quad x \in X.$$

Definition 2.4. A point $\bar{x} \in X$ is said to be efficient (Pareto), if there exists no $y \in X$ such that $f(y) \leq f(\bar{x})$.

Definition 2.5. A point $\bar{x} \in X$ is said to be weakly efficient, if there exists no $y \in X$ such that $f(y) < f(\bar{x})$.

3. Main results

In this section, using the concept of pseudo-α-invex functions, we shall extend the results given by Ruiz-Garzon et al. [12] for pseudo-invex functions.

In the following theorem we establish that under α-invexity assumptions the solutions of the generalized vector variational-like inequality problem (GVVLIP) are efficient solutions to (VOP).

Theorem 3.1. *Let $f : X \subset R^n \to R^p$ be differentiable function on X. If $F = \nabla f$, then f is α-invex with respect to α and η and $\bar{x}$ solves the generalized vector variational-like inequality problem* (*GVVLIP*) *with respect to the same α and η, then $\bar{x}$ is an efficient point to the vector optimization problem* (*VOP*)*.*

Proof. Suppose $\bar{x}$ is not an efficient point to (VOP), then there exists a $y \in X$ such that $f(y) - f(\bar{x}) \leq 0$.

Since f is α-invex with respect to α and η, we have ensured that $\exists y \in X$, such that

$$\big\langle \alpha(y,\bar{x})\nabla f(\bar{x}), \eta(y,\bar{x})\big\rangle \leqslant 0;$$

therefore $\bar{x}$ cannot be a solution to the generalized vector variational-like inequality problem (GVVLIP). This contradiction leads to the result. $\square$

Theorem 3.2. *Let $f : X \subset R^n \to R^p$ be differentiable function on X. If $F = \nabla f$, then $-f$ is strictly-α-invex with respect to α and η. If $\bar{x}$ is a weakly efficient solution to the* *vector optimization problem* (*VOP*) *then* $\bar{x}$ *also solves the generalized vector variational-like inequality problem* (*GVVLIP*).

Proof. Suppose that $\bar{x}$ is a weakly efficient solution to (VOP), but does not solve the (GVVLIP). Then there exists a $y \in X$ such that $\langle \alpha(y,\bar{x})\nabla f(\bar{x}), \eta(y,\bar{x})\rangle \leqslant 0$. By the strict-$\alpha$-invexity of $-f$ with respect to α and η, we have

$$f(y) - f(\bar{x}) < \left\langle a(y,\bar{x})\nabla f(\bar{x}), \eta(y,\bar{x})\right\rangle \leqslant 0;$$

therefore $\exists y \in X$ such that $f(y) < f(\bar{x})$, which contradicts $\bar{x}$ being a weakly efficient solution to the (VOP). $\square$

Since every efficient solution to (VOP) is a weakly efficient solution to (VOP), from the above theorem, we can get the following result.

Corollary 3.1. Let $f : X \subset R^n \to R^p$ be differentiable function on X. If $F = \nabla f$, then $-f$ is strictly-α-invex with respect to α and η. If $\bar{x}$ is an efficient solution to the vector optimization problem (VOP) then $\bar{x}$ also solves the generalized vector variational-like inequality problem (GVVLIP).

Theorem 3.3. *Let* $f : X \subset R^n \to R^p$ *be differentiable function on* X. *If* $F = \nabla f$, *and if* $\bar{x}$ *is a weakly efficient solution to the vector optimization problem* (*VOP*) *then* $\bar{x}$ *also solves the generalized weak vector variational-like inequality problem* (*GWVVLIP*).

If f *is a pseudo-*α*-invex function with respect to* α *and* η *and if* $\bar{x}$ *also solves the generalized weak vector variational-like inequality problem* (*GVVLIP*) *with respect to the same* α *and* η, *then* $\bar{x}$ *is a weakly efficient solution to the vector optimization problem* (*VOP*).

Proof. ($\Rightarrow$) Let $\bar{x}$ be a weakly efficient solution to the (VOP), since X is an α-invex set, we have that $\nexists y \in X$, such that $f(\bar{x} + t\alpha(y,\bar{x})\eta(y,\bar{x})) - f(\bar{x}) < 0$, $0 < t < 1$. Dividing the above inequality by t and taking the limit as $t \to 0$, we get to $\nexists y \in X$, such that $\langle \alpha(y,\bar{x})\nabla f(\bar{x}), \eta(y,\bar{x})\rangle < 0$.

($\Leftarrow$) If $\bar{x}$ is not a weakly efficient solution to (VOP), then $\exists y \in X$, such that

$$f(y) < f(\bar{x}).$$

By pseudo-α-invexity of f with respect to α and η, we have ensured that $\exists y \in X$, such that $\langle \alpha(y,\bar{x})\nabla f(\bar{x}), \eta(y,\bar{x})\rangle < 0$. This contradicts the fact that $\bar{x}$ is a solution to the (GWVVLIP). $\square$

Theorem 3.4. *Let* $f : X \subset R^n \to R^p$ *be differentiable function on* X. *If* $F = \nabla f$, f *is strictly-*α*-invex with respect to* α *and* η. *If* $\bar{x}$ *is a weakly efficient solution to the vector optimization problem* (*VOP*) *then* $\bar{x}$ *is an efficient solution to* (*VOP*).

Proof. Suppose that $\bar{x}$ is a weakly efficient solution to the (VOP), but not an efficient solution to (VOP). Then, there exists $\exists y \in X$, such that $f(y) \leqslant f(\bar{x})$. By the strict-$\alpha$-invexity of f with respect to the same α and η, we have

$$f(y) - f(\bar{x}) > \left\langle \alpha(y,\bar{x})\nabla f(\bar{x}), \eta(y,\bar{x})\right\rangle,$$

which is to say, $\exists y \in X$, such that $\langle \alpha(y, \bar{x}) \nabla f(\bar{x}), \eta(y, \bar{x}) \rangle < 0$; therefore, $\bar{x}$ does not solve the (GWVVLIP). This contradiction arises from the first part of Theorem 3.3. □

In the sequel we need the following definition from [10].

Definition 3.1. A feasible solution $\bar{x} \in X$ is said to be a vector critical point for the problem (VOP) if there exists a vector $\lambda \in R^p$ with $\lambda \geqslant 0$ such that $\lambda^{\mathrm{T}} \nabla f(\bar{x}) = 0$.

It should be noticed that scalar stationary points are those whose vector gradients are zero. For vector problems, the vector critical points are those such that there exists a non-negative linear combination of the gradient vectors of each component of objective function, valued at that point, equal to zero.

The following theorem is extension to the context of pseudo-α-invexity of [10, Theorem 2.2] for pseudo-invex case.

Theorem 3.5. *All vector critical points are weakly efficient solutions if and only if the vector function f is pseudo-α-invex on X.*

Proof. The proof follows from the proof of [10, Theorem 2.2] and the discussion as above in this paper. □

In light of Theorems 3.3 and 3.5 we could relate the vector critical points to the solutions of the weak vector variational-like inequality problem (GWVVLIP), with the following result.

Corollary 3.2. Suppose that $F = \nabla f$. If the objective function is pseudo-α-invex with respect to α and η, then the vector critical points, the weakly efficient points and the solutions of the generalized weak vector variational-like inequality problem (GWVVLIP) are equivalent.

4. Conclusions

In this paper, we have extended an earlier work of Ruiz-Garzon et al. [12] to a wider class of functions, namely α-invex and pseudo-α-invex functions. Furthermore, we have extended the results given by Ruiz-Garzon et al. [12], Yang and Chen [15] and Yang and Goh [16] from convex and pre-invex functions to pseudo-α-invex functions.

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